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Paper No. : Classical Mechanics
Module : Hamilton's Principle and Lagrange's Equation

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Description of Module	
Subject Name	Physics
Paper Name	Classical Mechanics
Module Name/Title	Hamilton’s Principle and Lagrange’s Equation
Module Id	5

Hamilton's Principle and Lagrange's Equation

Historical Note:

Historically a minimum principle is based on the notion that nature always acts in a way that during the development of a dynamical system, certain quantities are minimized. Fermat applied the principle of 'least time' that is light always travels from one point to another in the medium along a path least time to arrive at the laws of reflection and refraction. Manupertuis in 1747 made the first application of minimum principle in mechanics. A mathematical foundation of the principle was laid by Lagrange around 1760. Subsequently a more general principle in the form of Principle of Least Action was developed by Hamilton during 1834-35.

Contents:

1. Learning Objectives :

- You will learn about Hamilton's Principle of least action and its use in the derivation of Lagrange's Equation of Motion

- Solution of some simple examples will illustrate the power of the formulation.

1. Introduction:

Non-relativistic dynamics in an inertial frame are described by Newton's equation $\boldsymbol{F} = \dot{\boldsymbol{p}}$. In actual physical situation the dynamical system in general, is constrained by a prior unknown forces. A particle may be constrained to move on a given surface, the motion may be restricted within certain boundaries and so on. The constraint forces may be quite complicated in which case one may have to look for alternative formalism. This alternative formalism of course, cannot go beyond Newton's Laws but may result in the simplification of the problem and may even have wider application. Historically a minimum principle based on the notion **that nature always act in a way that during the development of a dynamical system, certain quantities are minimized,** has been used in an alternative formulation of mechanics. We will discuss in this unit a very powerful principle due to Hamilton for the formulation of what is known **Hamiltonian dynamics**.

2. Hamilton's Principle.

The principle states that **"of all the possible paths consistent with the constraints along which a dynamical system can evolve from one point to another, the actual path followed is the one which minimizes the action".** Where action is defined as the time integral of the Langrangian of the system i.e.

$$S = \int_{t_1}^{t_2} L dt \qquad (5.1)$$

Lagrange's Equation of motion

Consider a multiparticle system characterized by a Lagrangian function.

$$L = L(q_1, q_2, \dots \dots \dots \dots, \dot{q}_1, \dot{q}_2, \dots \dots \dots \dots ; t)$$
$$= L(\{q_k\}, \{\dot{q}_k\}; t) \qquad (5.2)$$

Where $\{q_k\}$ and $\{\dot{q}_k\}$ are the sets of generalized coordinate and velocities. The action for this Lagrangian is given as

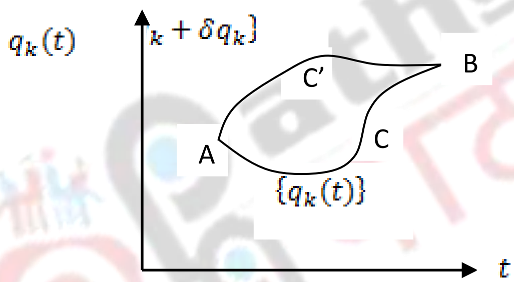
$$S = \int_{t_0}^{t} L(\{q_k\},\{\dot{q}_k\};t)\,dt \qquad (5.3)$$

Hamilton's Principle states that of the various paths given by
$q_1(t), q_2(t) \ldots \ldots \ldots \ldots \ldots \ldots . q_n(t), \dot{q}_1(t), \dot{q}_2(t) \ldots \ldots \ldots . \dot{q}_n(t)$

The path followed by the system is the one for which the corresponding action is minimum

Consider the path C specified by the set $\{q_k(t)\}$ and a nearby path C' characterized by the set $\{q_k + \delta q_k\}$.

The change δs in action is then given by



$$\delta s = \int_{t_0}^{t} \delta L(\{q_k\},\{\dot{q}_k\};t)\,dt$$

$$= \sum_{t_0}^{t} \int_{t_0}^{t} \left(\frac{\partial L}{\partial \dot{q}_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j \right) dt$$

Second term inside the integral can be integrated by parts

$$\int \frac{\partial L}{\partial \dot{q}_j}\delta\left(\frac{q_j}{dt}\right) dt = \int \frac{\partial L}{\partial \dot{q}_j}\frac{d}{\partial d}(\delta q_j)dt$$

$$= \frac{\partial L}{\partial \dot{q}_j}\,\delta q_j \Big|_{\delta q_j = 0} - \int \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j\;dt$$

The first term vanishes at the end points and we get

$$\delta s = \sum_{i} \int_{t_0}^{t} \left\{ \frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \right\} \delta q_j\;dt \qquad (5.5)$$

δs vanishes for arbitrary vaiations δq_j we thus obtain the Lagrange's equation of motion given by

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \quad ; \quad j = 1,2 \ldots \ldots \ldots . \qquad (5.6)$$

The Lagrangian as defined is not unique, we can add to it a term which is a total time derivative of any function $J(t)$ with no change in the equations of motion. The new action we get is

$$S = \int_{t_0}^{t} \left\{ L(\{q_k\},\{\dot{q}_k\};t) + \frac{dJ(t)}{dt} \right\} dt \qquad (5.7)$$

$$= \int_{t_0}^{t} \{L(\{q_k\},\{\dot{q}_k\};t)dt\} + \frac{dJ(t)}{dt}\bigg|_{t_0}^{t_1} dt$$

The last term being constant has no effect on δS and therefore, on Lagrange' equations.

Lagrange's Equation with Undetermined Multipliers:

In the above derivation we had assumed that the constraints are holonomic and can be expressed in terms of algebraic relations. If same of the constraints are expressed in terms of velocities and are in the form of non-integrable equations like

$$\sum_i A_i \dot{x}_i + B_i \dot{x}_i + C_i = 0$$

For arbitrary A_i's, B_i's and C_i's , it is possible to incorporate them in the Lagrange's eqations by means of the Lagrangian undermined multipliers. Take for example, non-holonomic constraints of the force

$$\int_l^c (q_i, \dot{q}_i, t) = 0 \qquad (5.8)$$

For constraints expressible as

$$\sum_j \frac{\partial \int_l^c}{\partial q_j} \delta q_j = 0$$

Where $j = 1,2, \dots \dots \dots \dots \dots \dots m \quad and \;\; l = 1,2, \dots \dots \dots \dots \dots \dots . S$

Where S is the number of constraints and q_j's are the generalized coordinates on which these constraints depend and the Lagrange's equation can be written as

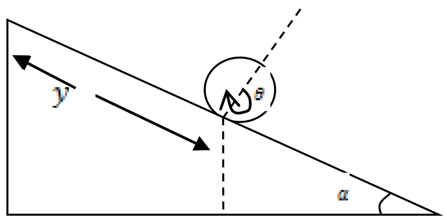
$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_l \lambda_l(t) \frac{\partial \int_l^c}{\partial q_j} = 0 \qquad (5.9)$$

Where q_j's are the undetermined multipliers.

3. Solved Examples:

1. Disc rolling down an inclined plane

A disc of mass M and radius R is rolling down an inclined planed without slipping write down the Lagrangian of the system and obtain the equation of motion.



The kinetic energy of the disc is the sum of translational and rotational energy

$$T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 \qquad (5.10)$$

Where I is the moment of inertia of the disc about its central axis.

The potential energy

$$U = Mg(l-y)\ \sin\alpha \qquad (5.11)$$

The potential energy is taken to be zero at the bottom of the plane

$$L = T - U = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 - Mg(l-y)\sin\alpha \qquad (5.12)$$

When the disc rotates by an angel θ, it tranverses a distance $R\theta$ along the plane, thus

$y = R\theta$ and we can write the Lagrangian as

$$L = \left(\frac{1}{2}M + \frac{1}{2}\frac{I}{R^2}\right)\dot{y}^2 + Mg(l-y)\sin\alpha \qquad (5.13)$$

Lagrange' equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial \dot{y}} = 0$$

$$\frac{d}{dt}\left(M + \frac{I}{R^2}\right)\dot{y} - Mg\sin\alpha = 0$$

$$\left(M + \frac{I}{R^2}\right)\ddot{y} = Mg\sin\alpha \qquad (5.14)$$

Acceleration is constant and is given by

$$\ddot{y} = \frac{g}{\left(M + \frac{I}{R^2}\right)}\sin\alpha \qquad (5.15)$$

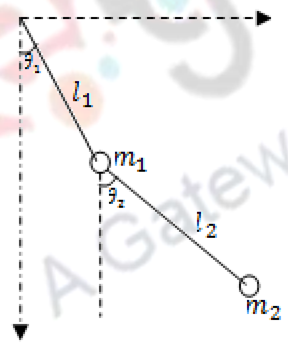
(Note that if instead of disc, we had a cylinder, a sphere, a ring or a spherical shell, the acceleration can be

obtained by putting the corresponding expression for the moment of inertia).

2. Double Pendulum:

For a double pendulum consisting of the masses m_1 and m_2 tied by string of length l_1 and l_2 supported at a point in the horizontal plane, obtain the Lagrangian and the equations of motion. Solve them for small amplitude.

Ans.: Set up the coordinate system as shown



The coordinate of the pendulum can be expressed in terms of angles θ_1 and θ_2 as

$$x_1 = l_1\cos\theta_1\,, y_1 = l_1\sin\theta_1 \qquad (5.16)$$

$x_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2 \;\; y_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$

$\dot{x}_1 = -l_1 \sin\theta_1 \dot{\theta}_1, \dot{y}_1 = l_1 \cos\theta_1 \dot{\theta}_1$

$\dot{x}_2 = -l_1 \sin\theta_1 \dot{\theta}_1 - l_2 \sin\theta_2 \dot{\theta}_2, \qquad \dot{y}_2 = l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \qquad (5.17)$$

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 \right)$$

$U = m_1 l_1 \cos\theta_1 - m_2 g + (l_1 \cos\theta_1 + l_2 \cos\theta_2)$

$= -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$

$L = T - U$

And the Lagrange' equation of motion are

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_2 + (m_1 + m_2) g l_1 \sin\theta_1 = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 + m_2 g l_2 \sin\theta_2 = 0 \qquad (5.18)$$

For $\theta_1, \theta_2 \ll 1$ and putting $m_1 = m_2 = m$ and $l_1 = l_2 = l$, we get

$$2\ddot{\theta}_1 + \ddot{\theta}_2 + \frac{2g}{l} \theta_1 = 0 \qquad (5.19)$$

$$\ddot{\theta}_2 + \ddot{\theta}_i + \frac{g}{l} \theta_2 = 0$$

The caxpled set of differential equations can be solved by assuming the solution to be

$$\theta_i A_i \cos\omega t \qquad (5.20)$$

Substituting in (4.17) we get

$$\left(\frac{2g}{l} - 2\omega^2 \right) A_1 - \omega^2 A_2 = 0 \qquad (5.21)$$

$$-\omega^2 A_1 - \left(\frac{g}{l} - \omega^2 \right) A_2 = 0$$

For the non-trivial solution to exist the determinant

$$\begin{vmatrix} \frac{2g}{l} - 2\omega^2 & -\omega^2 \\ -\omega^2 & \frac{2g}{l} - 2\omega^2 \end{vmatrix} = 0 \quad (5.22)$$

$$\omega^4 = \left(\frac{g}{l}\right)^2 (2 \pm \sqrt{2})$$

Thus we get the two frequencies.

3. Particle on a cycloid :

A particle is moving on a cycloid $\left(x = a(2\theta + \sin 2\theta)\right)$, $y = a(2\theta + \sin 2\theta)$ under the action of gravity set up the Lagrange's equation of motion and find the frequency for $\theta \ll 1$.

Solution: An infinitesimal distance element on the surface of the cycloid is

$$ds = \sqrt{dx^2 + dy^2}$$

$$= a\{(2d\theta + 2\cos 2\theta\, d\theta)^2 + (2d\theta + 2\sin 2\theta\, d\theta)^2\}^{1/2}$$

$$= 2a(2 + 2\cos 2\theta)^{1/2} d\theta = 4a \sin\theta$$

$$T = \frac{1}{2} m \left(\frac{ds}{dt}\right)^2 = 8ma^2 \cos^2\theta \dot{\theta}^2 \quad (5.23)$$

$$U = mgy = mga(1 - \cos 2\theta) \quad (5.24)$$

$$\therefore L = T - U = 8ma^2 \cos^2\theta \dot{\theta}^2 - mga(1 - \cos 2\theta) \quad (5.25)$$

And the Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$ gives

$$4a \cos^2\theta \ddot{\theta} + \frac{g}{l} \sin 2\theta = 0$$

For small oscillations

$$4a\ddot{\theta} = -2g\theta$$

$$\therefore \omega = \sqrt{\frac{g}{2a}} \qquad (5.26)$$

4. Summary

- Principle of least action is a very powerful method used for an alternative formulation of mechanics.
- Hamilton's Principle states that dynamical system evolves along a path that minimizes.